Classification of day-to-day dynamical models: a review

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Equilibrium and its stability

3.3. Stability

An equilibrium would be just an extreme state of rare occurrence if it were not stable — that is, if there were no forces which tended to restore equilibrium as soon as small deviations from it occurred.

Besides this stability "in the small" one may consider stability "in the large" — that is, the ability of the system to reach an equilibrium from any initial position. This latter type of stability is interesting not only because it concerns the capacity of the system to reach a new equilibrium position after some big change, but also because one may want to use an analogue of the adjustment process as a method of computing an equilibrium solution by successive approximations.

Beckmann, M., McGuire, C. B., & Winsten, C. B. (1956). *Studies in the Economics of Transportation*. New Haven: Yale University Press.

Stability and day-to-day adjustment process

The study of stability hinges ultimately on the question of how road users adjust themselves to changes — that is, how they adapt the extent of their travel by road and their choice of routes to varying traffic conditions. This, however, is one of the big unknowns of road-user behavior, so that at the present stage only conjectures are possible. Through a simple and plausible model one can get a rough picture of the minimum of conditions that must be met in order that the adjustment process should converge.

Beckmann, M., McGuire, C. B., & Winsten, C. B. (1956). *Studies in the Economics of Transportation*. New Haven: Yale University Press.

Past studies of day-to-day dynamics

Deterministic process (with continuous time scheme)

- Smith 1984
- Duouis and Nagurney 1993
- Friesz et al. 1994
- Smith and Wisten 1995
- Zhang and Nagurney 1996
- Zhang et al. 2001
- Mounce 2006
- Yang and Zhang 2009
- He et al. 2010
- Friesz et al. 2011

and others...

Stochastic process

- Horowitz 1984
- Cascetta and Cantarella 1993
- Cascetta 1989
- Cantarella and Cascetta 1995
- Watling 1999
- Watling and Hazelton, 2003
- Watling and Cantarella 2013
- Xiao and Lo 2014
- Cantarella and Watling 2015

and others...

Classifying Day-to-day models

- Day-to-day model should be investigated more to solve the problem on properties of equilibrium solution.
- How to build a model?
  - There are many ways to construct a day-to-day model.
  - Need to be classified so that we can understand how each model aims to describe actual human behaviour.

Autonomous system / myopic behaviour

State vector

$$\mathbf{x}^{(0)} \in X$$

$$\mathbf{x}^{(k)} = \boldsymbol{\mu}\left(\mathbf{x}^{(k-1)}\right)$$

$$(\boldsymbol{\mu}: X \to X; k = 1, 2, \ldots)$$

Time described as a discrete number

"…, we shall restrict attention to systems that are autonomous (the functional dependence on past states is time-invariant) and 1-dependent (the dependence may be written as a function of the immediately preceding states only,…)"

Watling (1999)

(highlighted by the speaker)

Autonomous system / myopic behaviour

- *"As an alternative to the equilibrium approach, we introduce an explicitly dynamic model in which each agent occasionally reconsiders his choice of strategy, using myopic rules to adjust his action in response to their current strategic environment. ..."*
- *"Myopia means that revisiting agent condition their choices on current behavior and payoff opportunities;"*

(Sandholm, 2010a)
(highlighted by the speaker)

Classifications (1)

Table 1
Classifying some possible convergent point behaviours for alternative types of dynamical systems of traffic assignment

	Continuous time	Discrete time
Deterministic process	Wardrop equilibrium/ stochastic equilibrium	Wardrop equilibrium/ stochastic equilibrium
Stochastic process	Stationary probability distribution	Stationary probability distribution

Watling (1999)

1. Continuous time or Discrete time?
2. Deterministic process ~~or Stochastic process?~~

Continuous time vs. Discrete time

$$\mathbf{x}(t+1) = \mathbf{f}\left(\mathbf{x}(t)\right)$$

Discrete time model

↓

$$\mathbf{x}(t+\alpha) = (1-\alpha)\mathbf{x}(t) + \alpha\mathbf{f}\left(\mathbf{x}(t)\right)$$

Exponential Smoothing

↓

$$\lim_{\alpha \to +0} \frac{\mathbf{x}(t+\alpha) - \mathbf{x}(t)}{\alpha} = \mathbf{f}\left(\mathbf{x}(t)\right) - \mathbf{x}(t)$$

↓

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left(\mathbf{x}(t)\right) - \mathbf{x}(t)$$

Continuous time model

General form of function f

$$\mathbf{x}(t) = (x_1(t), \ldots, x_i(t), \ldots, x_m(t))$$

# of users selecting choice i.

A: Choice set

$m = |A|$

$$f_a(\mathbf{x}(t)) = \sum_{b \in A} x_b(t)\rho_{ba} - \sum_{b \in A} x_a(t)\rho_{ab} + x_a(t)$$

# of users coming from other choices

# of users going to other choices

ρ_{ab} Proportion of users changing their choice from a to b. (depends on $\mathbf{x}(t)$ in some way)

General form of function *f*

$$\dot{x}_a(t) = \sum_{b \in A} x_b(t)\rho_{ba} - \sum_{b \in A} x_a(t)\rho_{ab}$$

ODE appears in Sandholm (2010ab)
(with minor modifications)

Next question: How to determine ρ_{ab}?

Classifications (2): Ways to set ρ_{ij}

by Sandholm (2010a)

1. Imitative Dynamics
   e.g. Replicator Dynamics
2. Excess Payoff Dynamics
   e.g. Brown-von Neumann-Nash (BNN) Dynamics
3. Pairwise Comparison Dynamics
   e.g. Smith Dynamics
4. Best Response Dynamics / Perturbed BR Dynamics
5. Projection Dynamics

1. Imitative dynamics

$$\rho_{ab} = x_b r_{ab}$$

Players tend to select a choice that is used by others.

e.g.)

Utility of strategy a

$$\rho_{ab} = x_b [u_b - u_a]_+$$

where $[x]_+ = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{otherwise} \end{cases}$

Replicator dynamics

Taylor, P., & Jonker (1978)
Schuster & Sigmund (1983)

2. Excess payoff dynamics

$$\rho_{ab} = \hat{\rho}_b \left(u_a - \bar{u} \right)$$

Weighted mean of utility for all players

players

$$\bar{u} = m^{-1} \sum_{a \in A} x_a u_a$$

e.g.)

$$\rho_{ab} = \left[u_a - \bar{u} \right]_+$$

Brown-von Neumann-Nash (BNN) dynamics

Brown & von Neumann (1950)
Nash (1951)

3. Pairwise comparison dynamics

$$\mathrm{sgn}(\rho_{ab}) = \mathrm{sgn}([u_b - u_a]_+)$$ 'Sign preserving'

Users move to strategy b only if doing so increases their new utility.

e.g.)

$$\rho_{ab} = [u_b - u_a]_+$$

Smith dynamics Smith (1984)

4. (Perturbed) Best response dynamics

$$\rho_{ab} = \sigma_b$$

Target protocols:

ρ_{ab} depends on a status of the target choice (i.e. b) only.

$$\rho_{ab} = \sigma_b = \arg\max_{\mathbf{y} \in \Delta} \sum_{a \in A} y_a u_a$$

Best response

(Gilboa and Matsui(1991))

$$\sigma_b = \frac{\exp(\theta u_b)}{\sum_{a \in A} \exp(\theta u_a)}$$

Perturbed best response

Logit type: Other types such as Probit can be formulated.

(Hofbauer & Sandholm (2007))

5. Projection dynamics

$$\rho_{ab} = \frac{[u_b - u_a]_+}{n x_a}$$

"… it describes the behaviour of agents who are uncomfortable playing strategies not used by others" (Sandholm, 2010a)

(Compare with the replicator dynamics $\rho_{ab} = x_b [u_b - u_a]_+$)

Classification of dynamics

Positive correlation

Table 5.1
Families of Evolutionary Dynamics and Their Properties

Family	Leading Example	(C)	$\leq$ (Dn)	(PC)	(NS)
Imitation	Replicator	yes	yes	yes	no
Excess payoff	BNN	yes	no	yes	yes
Pairwise comparison	Smith	yes	yes	yes	yes
	Best response	no	yes	yes[a]	yes[a]
Perturbed best response	Logit	yes	yes	no	no
	Projection	no	no	yes	yes

a. The best response dynamics satisfy versions of conditions (NS) and (PC) defined for differential inclusions.

Nash Stationary

Sandholm (2010a)

Global convergence: Potential game

- The combination of (C), (NS), and (PC) implies that the dynamics always converges to a Nash equilibrium point if the utility function can be associated with a potential function, i.e. there exist Φ satisfying $\mathbf{u}(\mathbf{x}) = \nabla\Phi(\mathbf{x})$.

(Potential game: Monderer, D., & Shapley, L. S., 1996)

Sandholm (2001)

Global convergence: Stable game

- The global convergence for any pairwise comparison model is also guaranteed if the utility function is monotonically decreasing, i.e.

$$\sum_{g\in G}\sum_{a\in A^g}(u_a^g(\mathbf{x})-u_a^g(\mathbf{y}))(x_a^g-y_a^g)\le 0 \quad \forall \mathbf{x},\mathbf{y}\in X$$

  – called ‘stable game’.

Hofbauer and Sandholm (2009)

More general cases?

- Local stability analysis
- Instable case?
  - Within-day dynamic traffic assignment cf. Iryo (2013)

Iryo, T. (2008). An Analysis of Instability in a Departure Time Choice Problem. *Journal of Advanced Transportation*, *42*(3), 333–356.

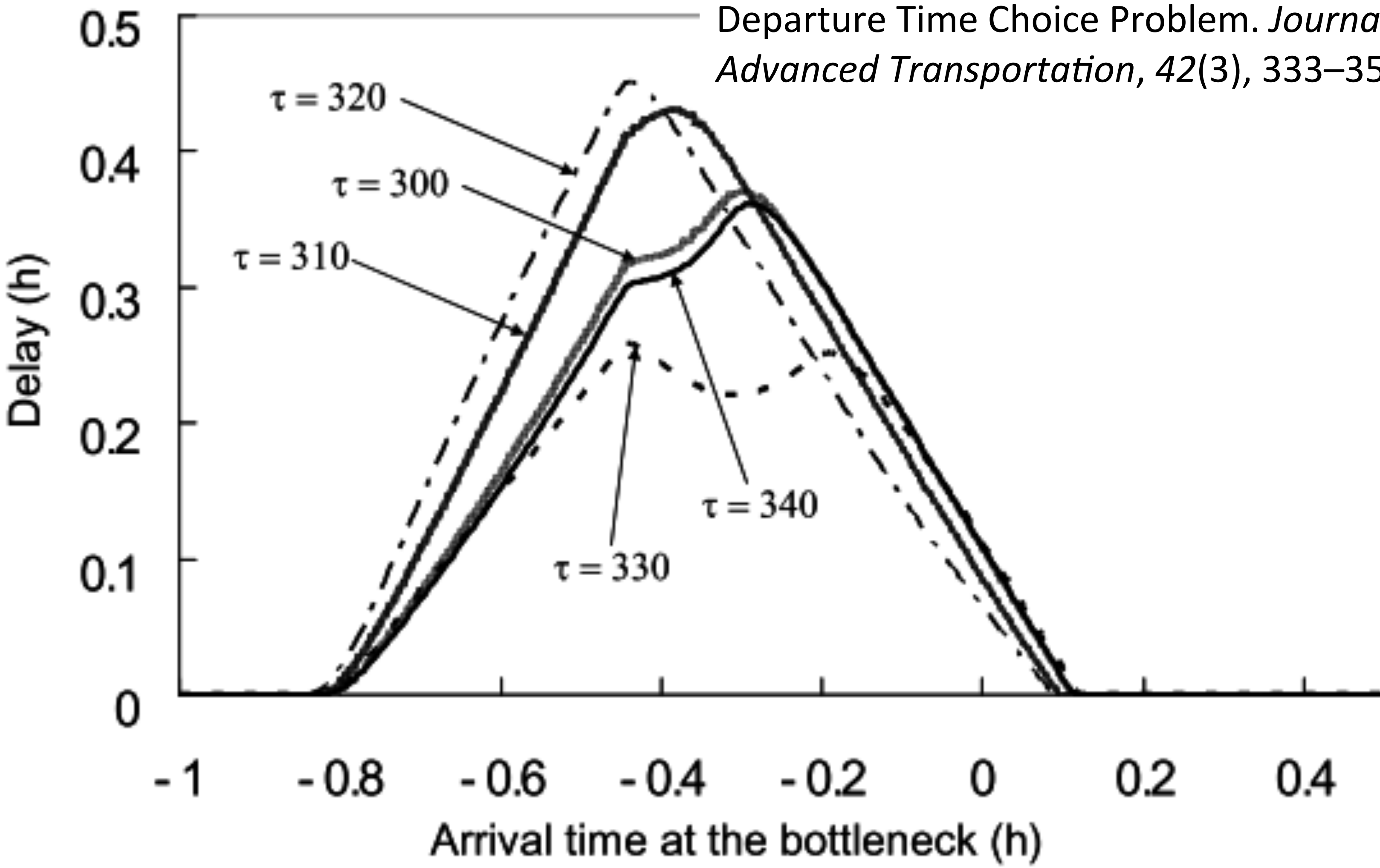

Figure 5. Bottleneck delay at different τ values when $\alpha = 0.5$ and $\beta = 5.0$

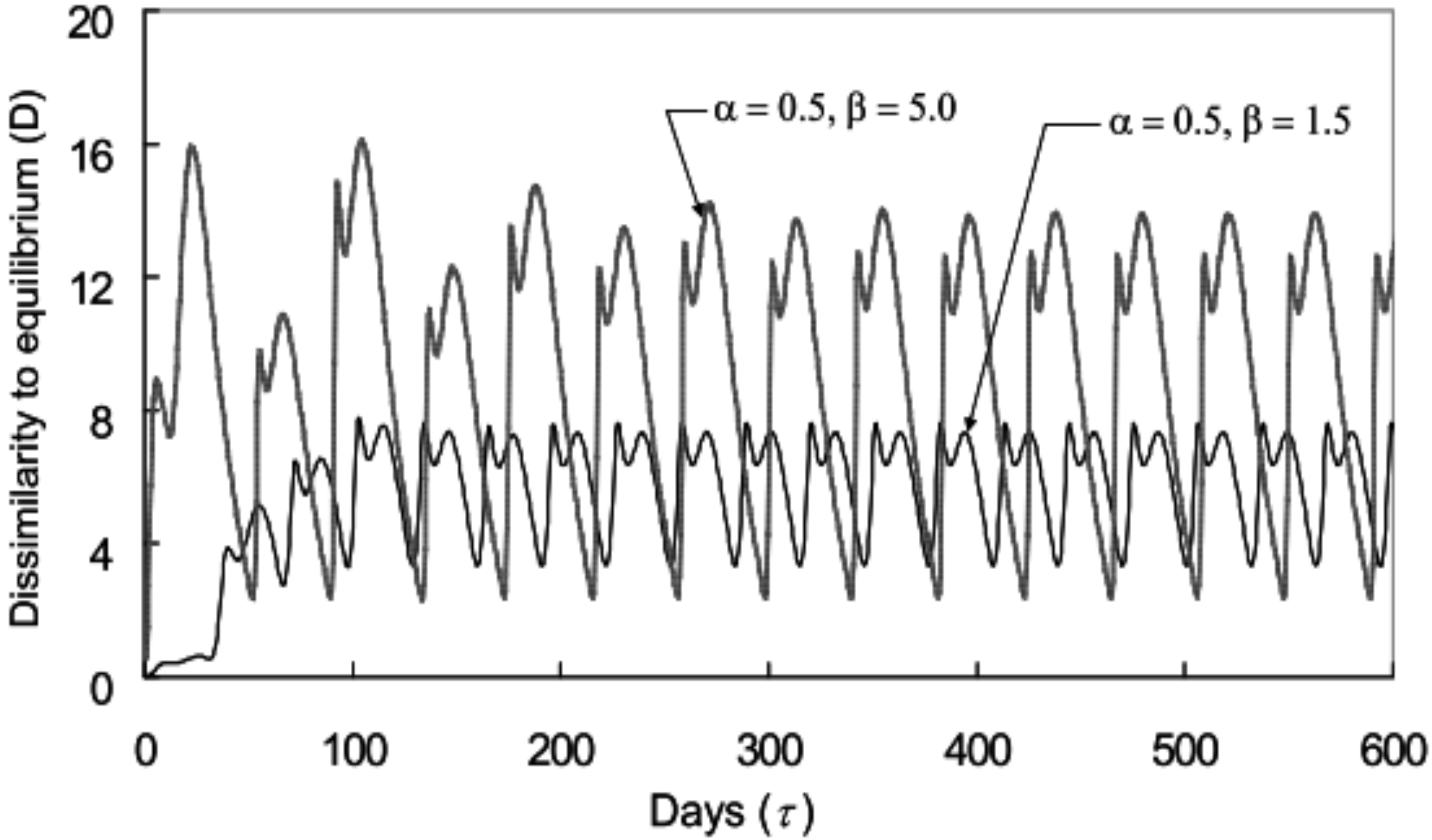

Figure 7. Change in $D\left(x^{\tau}\right)$ in the case where the equilibrium point received a perturbation at $\tau = 0$

Iryo, T. (2008). An Analysis of Instability in a Departure Time Choice Problem. *Journal of Advanced Transportation*, *42*(3), 333–356.

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